JOURNAL OF APPROXIMATION THEORY 41, 367–381 (1984)

Minimal Projections in Tensor-Product Spaces

C. FRANCHETTI*

Istituto di Matematica Applicata, Facoltà di Ingegneria, Università degli Studi di Firenze, 50139 Firenze, Italia

AND

E. W. CHENEY

Department of Mathematics, University of Texas, Austin, Texas 78712, U.S.A.

Communicated by Oved Shisha

Received July 3, 1983; revised October 14, 1983

1. INTRODUCTION

A *projection* of a Banach space X onto a subspace V is a bounded linear map $P: X \twoheadrightarrow V$ such that $P^2 = P$. (The arrow with two heads denotes a surjective map.) For many applications, a projection with nearly minimal norm is sought. The greatest lower bound for $\|P\|$ is the *relative projection constant* of V in X:

$$\lambda(V, X) = \inf\{\|P\|: P \in \mathscr{L}(X, V), P(X) = V, P^2 = P\}.$$

The *absolute projection constant* of a Banach space Y is defined by

$$\lambda(Y) = \sup\{\lambda(Y, Z): Z \supset Y\}.$$

These numbers may be infinite.

Our interest here is in the projection constants of subspaces of tensor-product spaces. For example, if $G \subset X$ and $H \subset Y$ (all Banach spaces), how is $\lambda(G \otimes H, X \otimes Y)$ related to $\lambda(G, X)$ and $\lambda(H, Y)$? This problem does not become properly posed until the topology of $X \otimes Y$ has been specified. It is convenient to assume that a *reasonable* norm α has been defined on the

* During the preparation of this paper Dr. Franchetti was a visiting professor at the University of Texas.

0021-9045/84 $3.00

algebraic tensor product $X \otimes Y$. This term means that in addition to the usual axioms of a norm, we require

$$\varepsilon(t) \leqslant \alpha(t) \leqslant \gamma(t), \qquad t \in X \otimes Y \tag{1}$$

$$\alpha|(A \otimes B)t| \leqslant \|A\| \|B\| \alpha(t), \qquad A \in \mathscr{L}(X, X_1), B \in \mathscr{L}(Y, Y_1). \tag{2}$$

In Eq. (1), ε denotes the injective tensor-product norm, also termed the "least cross-norm whose associate is also a cross norm" [13]. (In Schatten's monograph ε is denoted by λ, but here we wish to use λ for projection constants.) The norm γ is the projective tensor-product norm, or the "greatest cross-norm."

In the nomenclature of Diestel and Uhl [7], α is a reasonable cross-norm with the additional property (2). Our terminology agrees with that of Gilbert and Leih [8] except that we do not insist that α be defined for all pairs of Banach spaces X, Y.

The completion of $X \otimes Y$ with the norm α is denoted by $X \otimes_\alpha Y$. If $G \subset X$ and $H \subset Y$, we use the notation $G \mathbin{\bar{\otimes}_\alpha} H$ to denote the closure of $G \otimes H$ in $X \otimes_\alpha Y$. This may differ from $G \otimes_\alpha H$ in cases where α is of "general character" and therefore has a meaning for any pair of normed spaces. See [7, p. 231] and [13, p. 39].

Our main result is an extension of a recent theorem of Jameson and Pinkus [10]. They proved that if S and T are compact Hausdorff spaces, each containing infinitely many points, then the relative projection constant of $C(S) + C(T)$ as a subspace of $C(S \times T)$ is 3. Our result (Theorem 8) states that under the same hypotheses, and with G and H finite-dimensional subspaces containing the constant functions, the subspace

$$G \otimes_\varepsilon C(T) + C(S) \otimes_\varepsilon H$$

has relative projection constant at least 3. Our lower bound is sharp when $\lambda(G, C(S)) = \lambda(H, C(T)) = 1$, as occurs in the situation considered by Jameson and Pinkus.

Another of our results, Theorem 1, gives upper and lower bounds on the relative projection constant of $G \mathbin{\bar{\otimes}_\alpha} H$ as a subspace of $X \otimes_\alpha Y$. This theorem is accompanied by various examples which indicate that the upper and lower bounds can be attained.

2.

THEOREM 1. *Consider four Banach spaces, $G \subset X$, $H \subset Y$. Let α be a reasonable norm on $X \otimes Y$. The following inequality is valid for relative projection constants*:

$$\max\{\lambda(G, X), \lambda(H, Y)\} \leqslant \lambda(G \mathbin{\bar{\otimes}_\alpha} H, X \otimes_\alpha Y) \leqslant \lambda(G, X) \cdot \lambda(H, Y).$$

Proof. In order to prove the inequality on the right, let P and Q be projections of X onto G and Y onto H, respectively. Then $P \otimes_\alpha Q$ projects $X \otimes_\alpha Y$ onto $G \bar{\otimes}_\alpha H$ and has norm $\|P\| \cdot \|Q\|$. Hence

$$\lambda(G \bar{\otimes}_\alpha H, X \otimes_\alpha Y) \leqslant \|P\| \cdot \|Q\|.$$

By taking an infimum on P and Q we obtain the desired inequality.

In order to prove the inequality on the left, let P be a projection of $X \otimes_\alpha Y$ onto $G \bar{\otimes}_\alpha H$. Since the case $H = 0$ is trivial, we assume $H \neq 0$ and select $h \in H$ with $\|h\| = 1$. Select $\varphi \in Y^*$ such that $\|\varphi\| = 1$ and $\varphi(h) = 1$. Define $\tilde{\varphi}: X \otimes_\alpha Y \to X$ by putting at first

$$\tilde{\varphi}(x \otimes y) = \varphi(y) \cdot x \qquad (x \in X, y \in Y)$$

and then extending $\tilde{\varphi}$ by linearity and continuity. The continuous extension is possible because

$$\left\| \tilde{\varphi} \left(\sum x_i \otimes y_i \right) \right\| = \left\| \sum \varphi(y_i) \cdot x_i \right\| = \left\| \left(\sum x_i \otimes y_i \right) (\varphi) \right\|$$

$$\leqslant \varepsilon \left(\sum x_i \otimes y_i \right) \|\varphi\| \leqslant \alpha \left(\sum x_i \otimes y_i \right).$$

In this inequality, ε denotes the smallest reasonable norm (λ in Schatten's notation). Hence $\varepsilon \leqslant \alpha$. Also $\sum x_i \otimes y_i$ is interpreted as a linear operator from Y^* to X whose value at ψ is $\sum \psi(y_i) \cdot x_i$. The inequality then shows that $\|\tilde{\varphi}\|_\alpha \leqslant 1$, where $\|\tilde{\varphi}\|_\alpha$ is defined as the supremum of $\|\tilde{\varphi}(v)\|/\alpha(v)$, $v \in X \otimes_\alpha Y$.

Now define $Q: X \to G$ by putting $Qx = \tilde{\varphi}[P(x \otimes h)]$. It is easily seen that Q maps X into G, that $Qg = g$ for all $g \in G$, that Q is bounded, and that $\|Q\| \leqslant \|P\|_\alpha$. Hence $\|P\|_\alpha \geqslant \lambda(G, X)$. By taking an infimum on P we obtain $\lambda(G \bar{\otimes}_\alpha H, X \otimes_\alpha Y) \geqslant \lambda(G, X)$. By symmetry we obtain $\lambda(G \bar{\otimes}_\alpha H, X \otimes_\alpha Y) \geqslant \lambda(H, Y)$. This establishes the desired inequality. ∎

COROLLARY 1. *Let G, X, Y be Banach spaces, with $G \subset X$. Let α be a reasonable norm on $X \otimes Y$. Then the relative projection constants obey*

$$\lambda(G, X) = \lambda(G \bar{\otimes}_\alpha Y, X \otimes_\alpha Y).$$

COROLLARY 2. *If, in Theorem* 1, $\lambda(H, Y) = 1$ *then* (*with α as above*)

$$\lambda(G \bar{\otimes}_\alpha H, X \otimes_\alpha Y) = \lambda(G, X).$$

Now let S and T be compact Hausdorff spaces. If $G \subset X = C(S)$, $Y = C(T)$, and if ε is the smallest reasonable norm, then by a theorem of

Grothendieck [14] $X \otimes_\varepsilon Y = C(S \times T)$ and $G \otimes_\varepsilon C(T) = C(T, G)$. Also, it is clear that

$$\lambda(C(T), C(S \times T)) = 1.$$

Hence, from the first Corollary,

$$\lambda(G \otimes_\varepsilon C(T), C(S \times T)) = \lambda(G, C(S)).$$

Recall also that if $G \subset C(S)$ and $\dim G < \infty$, then

$$\lambda(G, C(S)) = \lambda(G).$$

The upper bound and the lower bound given in Theorem 1 can be attained in nontrivial examples, as will be indicated in some of the following results.

An operator L is said to satisfy Daugavet's equation if $\|I - L\| = 1 + \|L\|$. This is a property of compact operators in $C[0, 1]$ and $L_1[0, 1]$. See [5] and [1].

LEMMA 1. *Let K be a closed set which is not open in a compact Hausdorff space S. Let $R: C(S) \to C(K)$ be the restriction map, and let E be any bounded linear extension map from $C(K)$ to $C(S)$. Then the projection ER obeys "Daugavet's equation": $\|I - ER\| = 1 + \|ER\| = 1 + \|E\|$.*

Proof. It is clear that $\|I - ER\| \leqslant 1 + \|ER\|$. In order to prove that $\|I - ER\| \geqslant 1 + \|ER\|$ we distinguish two cases. First, suppose that $\|E\| > 1$. Let $1 < \rho < \|E\|$. Then there exists $y \in C(K)$ such that $\|y\| = 1$ and $\|Ey\| > \rho$. Select $\sigma \in S$ such that $|(Ey)(\sigma)| = \|Ey\|$. We can assume that $(Ey)(\sigma) = \|Ey\|$. By the Tietze Extension Theorem, there exists $x \in C(S)$ such that $Rx = y$, $x(\sigma) = -1$, and $\|x\| = 1$. Note that $\sigma \notin K$ since $(Ey)(\sigma) > 1$ while $(Ey)(s) \leqslant 1$ for $s \in K$. Now we have

$$\|ER - I\| \geqslant (ERx - x)(\sigma) = \|Ey\| + 1 > \rho + 1.$$

Since ρ was arbitrary between 1 and $\|E\|$, $\|ER - I\| \geqslant \|E\| + 1$. Observe that in this part of the proof we did not use the hypothesis that K is not open.

In the second case, assume that $\|E\| = 1$. There is a net $s_\alpha \in S \backslash K$ such that $\lim s_\alpha \in K$. By the Tietze Theorem, there exist functions $x_\alpha \in C(S)$ such that $Rx_\alpha = 1$, $x_\alpha(s_\alpha) = -1$, and $\|x_\alpha\| = 1$. Then, by continuity of $E1$,

$$\|ER - I\| \geqslant (ERx_\alpha - x_\alpha)(s_\alpha) = (E1)(s_\alpha) + 1 \to 2. \quad \blacksquare$$

If K is a closed subset of a compact Hausdorff space S, the extension constant of K in S is the number

$$\eta(K, S) = \inf\{\|E\|: E \text{ is a bounded linear extension map from } C(K) \text{ to } C(S)\}.$$

THEOREM 2. *Let K and S be as above, and let J be the ideal in C(S) of functions vanishing on K. Then*

$$\lambda(J, C(S)) = \begin{cases} 1 + \eta(K, S) & \text{if } K \text{ is not open} \\ 1 & \text{if } K \text{ is open.} \end{cases}$$

Moreover, there is a minimal projection on J if and only if there is a minimal extension map from C(K) to C(S).

Proof. If $P: C(S) \to J$ is a projection, then by Theorem 1 of [6], $P = I - ER$ for some bounded linear extension map. Here R is the restriction map from $C(S)$ to $C(K)$. Now use the preceding Lemma. If K is not open then $\|P\| = 1 + \|E\| \geqslant 1 + \eta(K, S)$, whence $\lambda(J, C(S)) \geqslant 1 + \eta(K, S)$. On the other hand, if an extension E is given then $I - ER$ is a projection on J. Hence $\|E\| = \|I - ER\| - 1 \geqslant \lambda(J, C(S)) - 1$, whence $\eta(K, S) \geqslant \lambda(J, C(S)) - 1$.

If K is open, a minimal extension is defined by $(Ex)(s) = x(s)$ for $s \in K$ and $(Ex)(s) = 0$ for $s \in S \backslash K$. A minimal projection is defined by $P = I - ER$. Both P and E are of norm 1.

The proof is completed by noting that in these arguments P is minimal if and only if E is minimal. ∎

COROLLARY 3. *If K is a closed set in a metric space S, and if J is the ideal in C(S) of functions vanishing on K, then* $\lambda(J, C(S)) = 1$ *or* 2 *depending on whether K is open or not. In both cases a minimal projection exists.*

Proof. By the Borsuk–Dugundji Theorem [14, p. 365], there exists a linear extension map of norm 1. The result now follows from Theorem 2. ∎

COROLLARY 4. *The set of all projection constants* $\lambda(J, C(S))$ *for S a compact Hausdorff space and J an ideal in C(S) is* $\{1\} \cup [2, \infty]$.

Proof. By Corollary 3, we get values $\lambda = 1$ or 2. By a theorem of Benyamini [2] all numbers in $[1, \infty]$ occur as values of $\eta(K, S)$. By Corollary 1, all numbers in $[2, \infty]$ occur as values of $\lambda(J, C(S))$. It is noteworthy that in Benyamini's theorem K can be fixed and taken to be the unit cell in a nonseparable Hilbert space, with its weak topology. ∎

Remark. In Benyamini's example, the extension constants are exact. The same is true for the examples of Corson and Lindenstrauss [4]. In all of these cases, the corresponding ideals possess minimal projections.

THEOREM 3. *Let S and T be metric spaces. Let* F_1 *and* F_2 *be closed sets in S and T, respectively, of which at least one has a nonempty boundary. Let*

G and H be the ideals corresponding to F_1 *and* F_2. *Then* $\max\{\lambda(G, C(S)), \lambda(H, C(T))\} = \lambda\{G \otimes_\varepsilon H, C(S \times T)\}$.

Proof. By the preceding results, $\max\{\lambda(G, C(S)), \lambda(H, C(T))\} = 2$. Now $G \otimes H$ is an ideal in $C(S \times T)$. Indeed, if $x \in C(S)$, $y \in C(T)$, $g \in G$, and $h \in H$ then

$$(x \otimes y) \cdot (g \otimes h) = (x \cdot g) \otimes (y \cdot h).$$

Since G and H are ideals, $x \cdot g \in G$ and $y \cdot h \in H$. By linearity and continuity we conclude that $zu \in G \otimes H$ if $z \in C(S \times T)$ and $u \in G \otimes H$. By Corollary 3, $\lambda(G \otimes_\varepsilon H, C(S \times T)) = 2$. ∎

Remark. $G \otimes H$ consists of all functions which vanish on

$$(F_1 \times T) \cup (S \times F_2).$$

THEOREM 4. *If G and H are finite-dimensional subspaces of* $C(S)$ *and* $C(T)$, *respectively, then*

$$\lambda(G \otimes_\varepsilon H, C(S) \otimes_\varepsilon C(T)) = \lambda(G, C(S)) \cdot \lambda(H, C(T)).$$

Proof. The steps in the proof are:

$$\begin{aligned}\lambda(G \otimes_\varepsilon H, C(S) \otimes_\varepsilon C(T)) &= \lambda(G \otimes_\varepsilon H, C(S \times T)) && (1)\\ &= \lambda(G \otimes_\varepsilon H) && (2)\\ &= \lambda(G)\, \lambda(H) && (3)\\ &= \lambda(G, C(S)) \cdot \lambda(H, C(T)). && (4)\end{aligned}$$

Step 1 uses the fact that $C(S) \otimes_\varepsilon C(T)$ is isometric to $C(S \times T)$ if $x \otimes y$ is identified with the function $x(s)\, y(t)$. Steps 2 and 4 use a remark made above. Step 3 utilizes a theorem from [15], which asserts that for any two finite-dimensional Banach spaces, $\lambda(E \otimes_\varepsilon F) = \lambda(E)\, \lambda(F)$. The proof of this theorem utilizes results in [9]. ∎

THEOREM 5. *Let S be a compact metric space and G an ideal in* $C(S)$ *such that* $\lambda[G, C(S)] = 2$. *Let H be any hyperplane in* (c_0). *Then*

$$\lambda[G \otimes_\varepsilon H, C(S) \otimes_\varepsilon (c)] < \lambda[G, C(S)]\, \lambda[H, (c)]. \qquad (1)$$

If $\lambda[H, (c_0)] = 1$ *then*

$$\lambda[G \otimes_\varepsilon H, C(S) \otimes_\varepsilon (c)] = \max\{\lambda[G, C(S)], \lambda[H, (c)]\}. \qquad (2)$$

Proof. The space (c), of all convergent sequences, is $C(T)$ when T is the set $\{0, 1/n\}_{n=1}^{\infty}$. By considering the composition of two projections we have (and here we write $\otimes$ in place of $\otimes_{\varepsilon}$)

$$\lambda[G \otimes H, C(S \times T)] \leqslant \lambda[G \otimes H, G \otimes (c_0)]\, \lambda[G \otimes (c_0), C(S \times T)]. \quad (3)$$

If G is the ideal of all functions in $C(S)$ which vanish on a certain closed set $K \subset S$, then $G \otimes (c_0)$ is the ideal of all functions in $C(S \times T)$ which vanish on $F \equiv (K \times T) \cup (S \times \{0\})$. Since $\lambda[G, C(S)] = 2$, K is not open, by Corollary 3, and hence F is not open. By Corollary 3 again,

$$\lambda[G \otimes (c_0), C(S \times T)] = 2. \quad (4)$$

By a theorem in [3], the relative projection constants of hyperplanes in (c_0) lie in the interval $|1, 2)$. Hence

$$\lambda[H, (c_0)] < 2. \quad (5)$$

By the lemma which follows,

$$\lambda[H, (c)] \geqslant 2. \quad (6)$$

By Corollary 1

$$\lambda[G \otimes H, G \otimes (c_0)] = \lambda[H, (c_0)]. \quad (7)$$

Now by combining (3), (4), (7), and (5) we see that

$$\lambda[G \otimes H, C(S \times T)] < 4. \quad (8)$$

By combining the hypothesis $\lambda[G, C(S)] = 2$ with Eq. (6) we see that

$$\lambda[G, C(S)]\, \lambda[H, (c)] \geqslant 4. \quad (9)$$

Thus (1) is established. In order to prove (2), we assume

$$\lambda[H, (c_0)] = 1. \quad (10)$$

Since $\lambda[G, C(S)] = 2 \leqslant \lambda[H, (c)]$, Theorem 1 implies that

$$\lambda[G \otimes H, C(S \times T)] \geqslant \max\{\lambda[G, C(S)], \lambda[H, (c)]\} = \lambda[H, (c)] \geqslant 2. \quad (11)$$

On the other hand, Eqs. (3), (7), (10), and (4) yield

$$\lambda[G \otimes H, C(S \times T)] \leqslant 2. \quad \blacksquare$$

LEMMA 2. *If H is a hyperplane in (c_0), then $\lambda[H, (c)] \geqslant 2$.*

Proof. Every projection $P: (c) \twoheadrightarrow H$ is of the form

$$Px = x - \langle \varphi, x \rangle z - \langle \psi, x \rangle w$$

with $\langle \varphi, x \rangle = \lim x_n$, $\psi \in (l_1)$, $\langle \varphi, z \rangle = \langle \psi, w \rangle = 1$, $\langle \varphi, w \rangle = \langle \psi, z \rangle = 0$, $H = \ker(\psi)$, $\|\psi\| = 1$.

Given $\varepsilon > 0$, select an integer k such that $|w_k| < \varepsilon$ and $z_k > 1 - \varepsilon$. Select $x \in (c)$ such that $\|x\| = 1$, $\lim x_n = -1$, and $x_k = 1$. Then

$$\begin{aligned} |(Px)_k| &= |x_k + z_k - \langle \psi, x \rangle w_k| \\ &\geqslant 1 + 1 - \varepsilon - \varepsilon = 2 - 2\varepsilon. \end{aligned}$$

Hence $\|P\| \geqslant \|Px\| \geqslant 2 - 2\varepsilon$. ∎

3.

In this section we study the projection constants of more complicated subspaces in tensor-product spaces. If $G \subset X$ and $H \subset Y$ are Banach spaces, and if α is a reasonable norm, we can define a subspace W of $X \otimes_\alpha Y$ by

$$W = \alpha\text{-closure in } X \otimes_\alpha Y \text{ of } (G \otimes Y) + (X \otimes H).$$

What can be learned about the relative projection constant of W as a subspace of $X \otimes_\alpha Y$?

THEOREM 6. *Let α be a reasonable norm on $X \otimes Y$. If both G and H are complemented subspaces, then so is W, and*

$$\lambda(W, X \otimes_\alpha Y) \leqslant \lambda(G, X) + \lambda(H, Y) + \lambda(G, X)\, \lambda(H, Y).$$

Proof. Let $P: X \twoheadrightarrow G$ and $Q: Y \twoheadrightarrow H$ be projections. Define a mapping L by

$$L = (P \otimes I_Y) \oplus (I_X \otimes Q).$$

Here we use the Boolean sum operation defined by $A \oplus B = A + B - AB$. This is a bounded linear operator on $X \otimes_\alpha Y$. It is routine to verify that L is a projection onto W, and that $\|L\| \leqslant \|P\| + \|Q\| + \|P\|\,\|Q\|$. ∎

THEOREM 7. *Let $A: X \to X$ and $B: Y \to Y$ be linear operators satisfying Daugavet's equation. Let α be a reasonable norm on $X \otimes Y$. Then the operator*

$$L = (A \otimes_\alpha I) \oplus (I \otimes_\alpha B)$$

on $X \otimes_\alpha Y$ also satisfies Daugavet's equation, and

$$\|L\| = \|A\| + \|B\| + \|A\| \, \|B\|.$$

Proof. By verifying that the two operators have the same effect on all dyads, $x \otimes y$, we obtain

$$I - L = (I - A) \otimes_\alpha (I - B).$$

From this and elementary results from [13, p. 30] we have

$$\begin{aligned} 1 + \|L\| \geqslant \|I - L\| &= \|I - A\| \, \|I - B\| \\ &= (1 + \|A\|)(1 + \|B\|) \\ &= 1 + \|A\| + \|B\| + \|A\| \, \|B\|. \end{aligned}$$

This proves "half" of our equation. The reverse inequality follows at once from the definition of L and the triangle inequality. ∎

THEOREM 8. *Let S and T be compact Hausdorff spaces, each containing infinitely many points. Let G and H be finite-dimensional subspaces containing the constants in $C(S)$ and $C(T)$, respectively. Then each projection of $C(S \times T)$ onto $G \otimes C(T) + C(S) \otimes H$ has norm at least 3.*

Proof. Let $n = \dim(G)$. Select $s_1, ..., s_n$ in S and $g_1, ..., g_n \in G$ so that $g_i(s_j) = \delta_{ij}$. Then the operator L defined by $Lx = \sum_{i=1}^n x(s_i) \, g_i$ is a projection of $C(S)$ onto G.

In the same way, let $m = \dim(H)$, and let M be a projection of $C(T)$ onto H of the form $My = \sum_{i=1}^m y(t_i) \, h_i$.

The operator $K = (I \otimes M) \oplus (L \otimes I)$ is a projection of $C(S \times T)$ onto the subspace $W = G \otimes C(T) + C(S) \otimes H$. Hence for any $w \in W$ we have $w = Kw$, or

$$\begin{aligned} w(s, t) = \sum_{\mu=1}^{n} w(s_\mu, t) \, g_\mu(s) + \sum_{\nu=1}^{m} w(s, t_\nu) \, h_\nu(t) \\ - \sum_{\mu=1}^{n} \sum_{\nu=1}^{m} w(s_\mu, t_\nu) \, g_\mu(s) \, h_\nu(t). \end{aligned} \tag{1}$$

Note that since $1 \in G$ and $1 \in H$, we have $L1 = 1$, $M1 = 1$, and

$$\sum_{\mu=1}^{n} g_\mu = 1, \qquad \sum_{\nu=1}^{m} h_\nu = 1. \tag{2}$$

Now let P be any projection of $C(S \times T)$ onto W. Let $\varepsilon > 0$. We will prove that $\|P\| > 3 - \varepsilon$.

Since S is compact and infinite, there exists an ω-accumulation point $\sigma \in S$. (See Kelley [11, p. 138].) Likewise, T contains an ω-accumulation point τ. Let c be any number greater than

$$2^n \max_{1 \leqslant i \leqslant n} \sum_{\mu=1}^{n} |g_\mu(\sigma) - g_\mu(s_i)| + 2^m \max_{1 \leqslant j \leqslant m} \sum_{\nu=1}^{m} |h_\nu(\tau) - h_\nu(t_j)|.$$

Let k be an integer so large that

$$k^{-2}[2k + 2kc\,\|P\| + (m+n)\,c + c^2\,\|P\|] < \varepsilon/2. \tag{3}$$

For $i = 1,\dots, k$ define a neighborhood of σ by

$$\mathscr{U}_i = \left\{ s \in S : \sum_{\mu=1}^{n} |g_\mu(\sigma) - g_\mu(s)| < c2^{-i} \right\}.$$

Note that for $i = 1,\dots, n$ we have $s_i \in \mathscr{U}_i$. Select inductively points $s_{n+1},\dots, s_k$ so that

$$s_1,\dots, s_n, s_{n+1},\dots, s_k \text{ are distinct} \tag{4}$$

$$s_i \in \mathscr{U}_i \qquad \text{for} \quad i = 1,\dots, k. \tag{5}$$

In the same way select points $t_{m+1},\dots, t_k$ so that

$$t_1,\dots, t_m, t_{m+1},\dots, t_k \text{ are distinct} \tag{6}$$

$$\sum_{\nu=1}^{m} |h_\nu(\tau) - h_\nu(t_j)| < c2^{-j} \qquad \text{for} \quad j = 1,\dots, k. \tag{7}$$

By an argument using partitions of unity, there exist $x_i \in C(S)$ such that $x_i \geqslant 0$, $x_i(s_j) = \delta_{ij}$, and $\sum_{i=1}^k x_i = 1$ for $1 \leqslant i, j \leqslant k$. Similarly, we have $y_j \in C(T)$ with $y_j \geqslant 0$, $y_j(t_i) = \delta_{ij}$, and $\sum_{j=1}^k y_j = 1$. Define $z_{ij} = x_i \otimes y_j$ $(1 \leqslant i, j \leqslant k)$. Elementary calculations show that

$$\|z_{ij}\| = 1 \tag{8}$$

$$z_{ij}(s_\mu, t_\nu) = \delta_{i\mu}\delta_{j\nu} \qquad (1 \leqslant i, j, \nu, \mu \leqslant k) \tag{9}$$

$$\sum_{i=1}^{k} z_{ij} = 1 \otimes y_j \in W \qquad (1 \leqslant j \leqslant k) \tag{10}$$

$$\sum_{j=1}^{k} z_{ij} = x_i \otimes 1 \in W \qquad (1 \leqslant i \leqslant k). \tag{11}$$

Define $w_{ij} = Pz_{ij}$. From Eqs. (10) and (11) we have

$$\|w_{ij}\| \leqslant \|P\| \tag{12}$$

$$\sum_{i=1}^{k} w_{ij} = 1 \otimes y_j \qquad (1 \leqslant j \leqslant k) \tag{13}$$

$$\sum_{j=1}^{k} w_{ij} = x_i \otimes 1 \qquad (1 \leqslant i \leqslant k). \tag{14}$$

Define

$$\varepsilon_{i\mu} = g_\mu(s_i) - g_\mu(\sigma) \qquad (1 \leqslant \mu \leqslant n,\ 1 \leqslant i \leqslant k) \tag{15}$$

$$\varepsilon'_{j\nu} = h_\nu(t_j) - h_\nu(\tau) \qquad (1 \leqslant \nu \leqslant m,\ 1 \leqslant j \leqslant k). \tag{16}$$

From Eqs. (5) and (7) we have

$$\sum_{\mu=1}^{n} |\varepsilon_{i\mu}| \leqslant c2^{-i} \qquad \text{and} \qquad \sum_{\nu=1}^{m} |\varepsilon'_{j\nu}| \leqslant c2^{-j}. \tag{17}$$

From Eq. (1) we have

$$\sum_{ij} w_{ij}(s_i, t_j) = \sum_{ij\mu} g_\mu(s_i)\, w_{ij}(s_\mu, t_j) + \sum_{ij\nu} h_\nu(t_j)\, w_{ij}(s_i, t_\nu) - \sum_{ij\mu\nu} g_\mu(s_i)\, h_\nu(t_j)\, w_{ij}(s_\mu, t_\nu). \tag{18}$$

The terms on the right side of Eq. (18) are now to be estimated from above. We have, using Eqs. (15), (13), (12), (2), and (17),

$$\begin{aligned}
\sum_{ij\mu} g_\mu(s_i)\, w_{ij}(s_\mu, t_j) &= \sum_{ij\mu} [g_\mu(\sigma) + \varepsilon_{i\mu}]\, w_{ij}(s_\mu, t_j) \\
&\leqslant \sum_{j\mu} g_\mu(\sigma)(1 \otimes y_j)(s_\mu, t_j) + \sum_{ij\mu} |\varepsilon_{i\mu}|\, \|P\| \\
&= \sum_{j\mu} g_\mu(\sigma) + k\,\|P\| \sum_{i\mu} |\varepsilon_{i\mu}| \\
&= \sum_{j} 1 + k\,\|P\| \sum_{i} c2^{-i} \\
&= k + kc\,\|P\|.
\end{aligned}$$

Similarly, the second term is bounded from above by $k + kc\,\|P\|$. The third term in Eq. (7) is estimated as follows:

$$\begin{aligned}
-\sum_{ij\mu\nu} & w_{ij}(s_\mu, t_\nu)\, g_\mu(s_i)\, h_\nu(t_j) \\
&= -\sum_{ij\mu\nu} [g_\mu(\sigma) + \varepsilon_{i\mu}][h_\nu(\tau) + \varepsilon'_{j\nu}]\, w_{ij}(s_\mu, t_\nu) \\
&= -\sum_{ij\mu\nu} [g_\mu(\sigma)\, h_\nu(\tau)(1 \otimes y_j)(s_\mu, t_\nu) + \varepsilon_{i\mu} h_\nu(\tau)\, w_{ij}(s_\mu, t_\nu) \\
&\quad + \varepsilon_{j\nu}\, g_\mu(\sigma)\, w_{ij}(s_\mu, t_\nu) + \varepsilon_{i\mu} \varepsilon'_{j\nu} w_{ij}(s_\mu, t_\nu)] \\
&= -1 - \sum_\mu \varepsilon_{\mu\mu} - \sum_\nu \varepsilon'_{\nu\nu} - \sum_{ij\mu\nu} \varepsilon_{i\mu} \varepsilon'_{j\nu} w_{ij}(s_\mu, t_\nu) \\
&\leqslant \sum_\mu |\varepsilon_{\mu\mu}| + \sum_\nu |\varepsilon'_{\nu\nu}| + \|P\| \sum_{ij\mu\nu} |\varepsilon_{i\mu} \varepsilon'_{j\nu}| \\
&\leqslant nc + mc + c^2 \|P\|.
\end{aligned}$$

When these estimates are combined, we have

$$\sum_{ij} w_{ij}(s_i, t_j) \leqslant 2k + 2kc\|P\| + (m+n)\, c + c^2 \|P\| \equiv A.$$

It follows, with the help of Eq. (3), that

$$\min_{ij} w_{ij}(s_i, t_j) \leqslant k^{-2} A < \varepsilon/2.$$

The proof is completed by the following calculation:

$$\begin{aligned}
0 &\leqslant 2(1 - x_i)(1 - y_j) \leqslant 2 \\
-1 &\leqslant 1 - 2x_i - 2y_j + 2x_i y_j \leqslant 1 \\
\|1 - 2x_i - 2y_j + 2x_i y_j\| &\leqslant 1 \\
\|P(1 - 2x_i - 2y_j + 2x_i y_j)\| &\leqslant \|P\| \\
\|1 - 2x_i - 2y_j + 2w_{ij}\| &\leqslant \|P\| \\
|1 - 2x_i(s_i) - 2y_j(t_j) + 2w_{ij}(s_i, t_j)| &\leqslant \|P\| \\
|-3 + 2w_{ij}(s_i, t_j)| &\leqslant \|P\| \\
3 - 2w_{ij}(s_i, t_j) &\leqslant \|P\| \\
3 - \varepsilon &\leqslant \|P\|. \quad \blacksquare
\end{aligned}$$

COROLLARY 5. *If, in Theorem* 8, *G and H possess norm-*1 *projections, then* $\lambda[W, C(S \times T)] = 3$. *Moreover, if P and Q are two norm-*1 *projections, then*

$$(P \otimes I) \oplus (I \otimes Q)$$

is a minimal projection onto W.

Proof. By Theorem 8,

$$\lambda[W, C(S \times T)] \geqslant 3.$$

It is easy to verify that the Boolean sum projection has norm at most 3. ∎

EXAMPLE. The case $W = C(S) + C(T)$ in Corollary 5 was first given by Jameson and Pinkus [10]. Their methods have been used in the proof of Theorem 8.

EXAMPLE. If $S = T = [0, 1]$, and if G and H are spaces of piecewise linear functions with prescribed knots, then $\lambda(W, C(S \times T)) = 3$.

Remark. In Theorem 8, if we drop the hypothesis that the subspaces G and H contain constants but assume instead the existence of $g \in G$ and $h \in H$ such that

$$\|g\| = \|h\| = 1, \qquad \alpha = \inf g(s) > 0, \qquad \beta = \inf h(t) > 0,$$

then each projection on W has norm at least $3\alpha\beta$. The proof is almost the same. The functions z_{ij} in the proof would be defined as $gx_i \otimes hy_j$, and at the end of the proof we would have the inequality

$$3g(s_i)\, h(t_j) - 2w_{ij}(s_i, t_j) \leqslant \|P\|.$$

THEOREM 9. *Let G be a subspace of $C(S)$ with relative projection constant λ_1. Let H be a subspace of finite codimension in $C(T)$ with relative projection constant λ_2. Assume that T has no isolated points. Let $W = C(S) \otimes H + G \otimes C(T)$. Then the relative projection constant of W as a subspace of $C(S \times T)$ does not exceed $\lambda_1(\lambda_2 - 1) + \lambda_2$.*

Proof. Let P and Q be projections of $C(S)$ and $C(T)$ onto G and H, respectively. Define $V = \ker(Q)$, $Q_2 = I - Q$, and $L = (P \otimes Q_2) + (I \otimes Q)$.

It is easily proved that $W = [G \otimes V] \oplus [C(S) \otimes H]$.

Now we prove that L maps $C(S \times T)$ into W. For any $z \in C(S \times T)$, $(P \otimes Q_2)\, z \in G \otimes V$ and $(I \otimes Q)\, z \in C(S) \otimes H$. Hence $Lz \in W$.

Next we prove that $Lw = w$ for any $w \in W$. If $g \in G$ and $v \in V$ then $L(g \otimes v) = (Pg \otimes Q_2 v) + (Ig \otimes Qv) = (g \otimes v) + (g \otimes 0) = g \otimes v$. By linearity and continuity, $Lz = z$ for all $z \in G \otimes V$. If $x \in C(S)$ and $h \in H$, then $L(x \otimes h) = (Px \otimes Q_2 h) + (Ix \otimes Qh) = (Px \otimes 0) + (x \otimes h) = x \otimes h$. By linearity and continuity, $Lz = z$ for all $z \in G \otimes V$. Hence $Lw = w$ for all $w \in W$.

Since Q_2 is compact, Daugavet's Theorem implies that $\|I - Q_2\| = 1 + \|Q_2\|$. Thus $\|Q\| = 1 + \|Q_2\|$.

From the definition of L we have at once $\|L\| \leqslant \|P\|\, \|Q_2\| + \|Q\| =$

$\|P\|(\|Q\|-1)+\|Q\|$. This number is then an upper bound for the projection constant of W. By taking an infimum on P and Q we arrive at the upper bound $\lambda_1(\lambda_2-1)+\lambda_2$. ∎

Open Problems

1. If G and H are finite-dimensional subspaces in $C(S)$ and $C(T)$, respectively, is the following equation necessarily true?

$$\lambda(G \otimes C(T) + C(S) \otimes H, C(S \times T)) = \lambda(G) + \lambda(H) + \lambda(G)\,\lambda(H).$$

2. If G and H are as in Question 1, does there necessarily exist a minimal projection of $C(S \times T)$ onto the subspace $W = G \otimes C(T) + C(S) \otimes H$? (By Corollary 5, the answer is affirmative when $\lambda(G) = \lambda(H) = 1$.)

3. Let G, H, W be as in Question 2. Assume that both S and T are infinite sets and that $\max\{\lambda(G), \lambda(H)\} > 1$. Does it necessarily follow that $\lambda(W, C(S \times T)) > 3$?

4. In Theorem 8, can we drop the hypothesis that G and H contain the constant functions?

Acknowledgments

We are grateful to Professor Dan Amir for many discussions concerning these matters and to Professors Jameson and Pinkus for their constructive comments on our first draft.

References

1. V. F. Babenko and S. A. Pichugov, A property of compact operators in the space of integrable functions, *Ukrain. Mat. Zh.* **33** (1981), 491–492. Transl.: Plenum MR82m: 47018.
2. Y. Benyamini, Constants of simultaneous extension of continuous functions, *Israel J. Math.* **16** (1973), 258–262.
3. J. Blatter and E. W. Cheney, Minimal projections on hyperplanes in sequence spaces, *Ann. Mat. Pura Appl.* **101** (1974), 215–227.
4. H. H. Corson and J. Lindenstrauss, On simultaneous extensions of continuous functions, *Bull. Amer. Math. Soc.* **71** (1965), 542–545.
5. I. K. Daugavet, A property of compact operators in the space C, *Uspehi Mat. Nauk* **18** (1963), 157–158. MR28-431.
6. D. W. Dean, Subspaces of $C(H)$ which are direct factors of $C(H)$, *Proc. Amer. Math. Soc.* **16** (1965), 237–242.
7. J. Diestel and J. J. Uhl, "Vector Measures," Amer. Math. Soc., Providence, R.I., 1977.
8. J. E. Gilbert and T. J. Leih, Factorization, tensor products, and bilinear forms in Banach space theory, *in* "Notes in Banach Spaces" (H. E. Lacey, Ed.), pp. 182–305, Univ. of Texas Press, Austin, 1980.

9. J. R. HOLUB, Tensor-product mappings, *Math. Ann.* **188** (1970), 1–12.
10. G. J. O. JAMESON AND A. PINKUS, Positive and minimal projections in function spaces, *J. Approx. Theory* **37** (1983), 182–195.
11. J. L. KELLEY, "General Topology," Van Nostrand, Princeton, N.J., 1955.
12. W. RUDIN, "Real and Complex Analysis," 2nd ed., McGraw–Hill, New York, 1974.
13. R. SCHATTEN, "A Theory of Cross-Spaces," Annals of Math. Studies, No. 26, Princeton Univ. Press, Princeton, N.J., 1950.
14. Z. SEMADENI, "Banach Spaces of Continuous Functions," Polish Scientific Publishers, Warsaw, 1971.
15. N. TOMCZAK-JAEGERMANN, Finite-dimensional operator ideals and the Banach–Mazur Theorem, *in* "Lecture Notes in Mathematics," Springer-Verlag, New York/Berlin, in press.